On phantom thermodynamics with negative temperature

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Abstract

We discuss the thermodynamic properties of the Friedmann-Robertson-Walker universe with dark energy fluids labelled by $\omega = p/\rho < -1/3$. Using the integrability condition, we show that the phantom phase of $w < -1$ can still be thermodynamically allowed even when the temperature takes on negative values because in that case, there exists at least a condition of keeping physical values for p and ρ.

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In the present and future universe, the study of phantom regime is a very important subject, since a phantom fluid of $p+\rho<0$ violates the strong and dominant energy conditions. Further, the energy density of a phantom field increases along the cosmic evolution thereby causing a super accelerating universe which will end in big rip [1, 2]. The big rip corresponds to the case that the singularity of $\rho \to \infty$ will appear at a finite time in the future [3].

The thermodynamics of phantom fields is considered within the framework of a dark fluid model based on the Euler's relation of $Ts = (1+\omega)\rho - n\mu$, where T, s, n, and μ are the temperature, entropy density, particle density, and chemical potential, respectively. In the case of classical fields, there is no relation to define the particle density n as a function of scalar field ϕ and its derivative $\dot{\phi}$. Hence, we choose the $\mu = 0$ case for simplicity. The phantom phase of $\omega < -1$ implies simply either $T > 0,\ s < 0$ or $T < 0,\ s > 0$.

In the first approach to phantom thermodynamics, a group of authors [4, 5] have insisted that the temperature of any dark energy component is always positive definite obeying the evolution law $T \sim a^{-3\omega}$ where $a(t)$ is the scale factor. In this case, they have shown that the existence of phantom fluids is not thermodynamically allowed because its co-moving entropy of $S \sim (1+\omega)T^{1/\omega}a^3$ is negative.

In the second approach [6], the authors have claimed that the temperature of phantom fluids in the FRW universe is negative as $T \sim (1+\omega)a^{-3\omega}$ for $\omega < -1$. If one accepts this temperature reinterpretation, it is possible to have the positive entropy of the phantom fields.

In this Letter, we derive pressure and energy density as functions of temperature: $p(T) = C(\omega)T^{(1+\omega)/\omega}$ and $\rho(T) = \tilde{C}(\omega)T^{(1+\omega)/\omega}$ with unknown functions $C(\omega)$ and $\tilde{C}(\omega)$. We show that for $C(\omega) \sim \omega$ and $\tilde{C}(\omega) \sim 1$, the second approach to phantom thermodynamics with negative temperature is not thermodynamically allowed since the internal inconsistency between thermodynamic quantities is arisen from the integrability condition. However, a specific choice of $C = (1+w)^{2n-(1+w)/w}$ and $\tilde{C} = C/w$ with $n = 0, 1, 2, \cdots$ leads to a phantom thermodynamics because $p(T)/\rho(T) = \omega$ provides a correct equation of state for $\omega < -1$. This implies that a phantom thermodynamics with negative temperature is still regarded as a thermodynamically meaningful system.

We start with the homogeneous and isotropic FRW universe which is described by two

Friedmann equations based on the Robertson-Walker metric

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{1}$$

$$\dot{H} = -4\pi G(\rho + p) + \frac{k}{a^2} \tag{2}$$

where $H = \dot{a}/a$ is the Hubble parameter and $k = -1, 0, 1$ represent the three-dimensional space with the negative, zero, and positive spatial curvature, respectively.

For the thermodynamic study, we apply the combination of the first- and second-law of thermodynamics to a comoving volume element of unit coordinate volume and physical volume $V = a^3$. Then it leads to

$$TdS = d(\rho V) + pdV = d[(\rho + p)V] - Vdp. \tag{3}$$

The integrability condition is necessary to define the FRW universe as a thermodynamic system [7]. It is given by

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T} \tag{4}$$

which leads to the relation between the pressure (energy density) and temperature

$$dp = \frac{\rho + p}{T} dT. \tag{5}$$

Plugging Eq.(5) into Eq.(3), we have the differential relation,

$$dS = \frac{1}{T} d[(\rho + p)V] - (\rho + p)V \frac{dT}{T^2} = d\left[\frac{(\rho + p)V}{T} + C\right] \tag{6}$$

where C is a constant. The entropy per comoving volume must be defined by

$$S \equiv \frac{(\rho + p)}{T} V \tag{7}$$

up to an additive constant.

On the other hand, the conservation law plays an important role in the FRW universe. This could be derived from Eqs.(1) and (2) as

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{8}$$

Hence, one equation among three is redundant. Here we choose the first Friedmann equations (1) and the conservation law (8) as the relevant equations. Importantly, Eq.(8) gives the energy density with $p = \omega\rho$

$$\rho(a) = \rho_0 a^{-3(1+\omega)}. \tag{9}$$

Considering the equivalent relation of $d(\rho V) + pdV = 0$, the conservation law corresponds to $dS = 0$. That is, the dark fluid should expand adiabatically as

$$d\left[\frac{(\rho+p)V}{T}\right] = 0, \tag{10}$$

which means that the entropy S per comoving volume is conserved. In other words, the FRW universe satisfying the conservation law should expand adiabatically.

Even for an adiabatic process of $S = \text{const}$, the same definition of entropy follows from the conservation law which can be rewritten as

$$d[(\rho+p)V] = V dp. \tag{11}$$

Inserting the integrability condition Eq.(5) into Eq.(11), one recovers Eq.(10) immediately.

Using $p = \omega\rho$, we rewrite the entropy of Eq.(7) as

$$S = \frac{(1+\omega)\rho V}{T}, \tag{12}$$

which defines the entropy in terms of the temperature and energy density. At this stage, we would like mention a work on the holographic approach to the FRW universe [8]. This author assumed that Eq. (12) defines the temperature in terms of the constant entropy as

$$T = \frac{(1+\omega)\rho_0}{S} a^{-3\omega}. \tag{13}$$

At that time, one implicitly believed that the scaling law $T \sim a^{-3\omega}$ holds for $\omega \geq -1$ because the literatures mainly focused on the holographic description for a radiation-dominated universe with $\omega = 1/3$ [9]. Later on, Eq.(13) was used for giving the origin to the idea of negative temperature for $\omega < -1$ and $S > 0$ [6]. Since then, an extension to $\omega < -1$ has been widely considered in many different contexts [10].

However, we wish to show that this extension may induce a main flaw when considering an evolution of the FRW universe with dark fluid as an adiabatic process of thermodynamic system. From the integrability condition Eq.(5), we have

$$\frac{dp}{p} = \left[\frac{1+\omega}{\omega}\right]\frac{dT}{T} \tag{14}$$

which leads to

$$\ln p = \left[\frac{1+\omega}{\omega}\right]\ln T + \ln C(\omega). \tag{15}$$

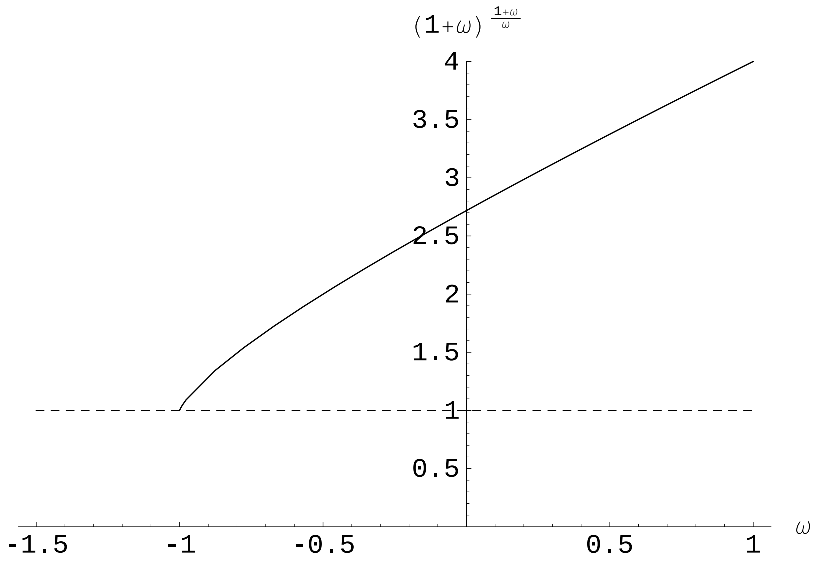

FIG. 1: Graph of $(1+\omega)^{\frac{1+\omega}{\omega}}$ as function of ω. The dotted line denotes "1".

In this case, we obtain the relation between pressure and temperature[1]

$$p(T) = C(\omega)T^{\frac{1+\omega}{\omega}} \tag{16}$$

with an unknown function $C(\omega)$. Also, from the other form of integrability condition

$$\frac{d\rho}{\rho} = \Big[\frac{1+\omega}{\omega}\Big]\frac{dT}{T} \tag{17}$$

we derive the generalized Stefan-Boltzmann law[2] [5]

$$\rho(T) = \tilde{C}(\omega)T^{\frac{1+\omega}{\omega}} \tag{18}$$

with an unknown function $\tilde{C}(\omega)$. At the first glance, it seems that one recovers the important relation of $p \sim \omega\rho$ if $C(\omega) \sim \omega$ and $\tilde{C}(\omega) \sim 1$. Then, substituting T in Eq.(13) into Eq.(16) leads to

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[1] As a simple example, we consider an ideal gas with constant heat capacities undergoing a reversible, adiabatic compression and expansion. In this case, we have an adiabatic process $dU + pdV = 0$ with $dU = C_V dT$, which leads to $C_V dT = -pdV$. Using its equation of state $pV = RT$ which implies $dV = -\frac{RT}{p^2}dp + \frac{R}{p}dT$, one finds $C_V dT = RT\frac{dp}{p} - RdT$. Considering $C_V + R = C_p$, it leads to $C_p dT = RT\frac{dp}{p}$ which can be rewritten as $\frac{dp}{p} = \Big[\frac{\gamma}{\gamma-1}\Big]\frac{dT}{T}$ with $\gamma = C_p/C_V$. Integration gives a similar formula for an adiabatic process of an ideal gas as $p = c(\gamma)T^{\frac{\gamma}{\gamma-1}}$.

[2] Actually, this expression differs from $\rho = \rho_0\Big(T/\kappa(1+\omega)\Big)^{\frac{1+\omega}{\omega}}$ in [6], which was obtained by plugging Eq.(13) into Eq.(9). In the case of $\omega < -1$, they must take $T < 0$ to preserve ρ positive.

$$p(T) = \omega(1+\omega)^{\frac{1+\omega}{\omega}} \left[\frac{\rho_0}{S}\right]^{\frac{1+\omega}{\omega}} a^{-3(1+\omega)}. \tag{19}$$

Similarly, plugging T into Eq.(18) leads to

$$\rho(T) = (1+\omega)^{\frac{1+\omega}{\omega}} \left[\frac{\rho_0}{S}\right]^{\frac{1+\omega}{\omega}} a^{-3(1+\omega)}. \tag{20}$$

Here we observe that if $\omega < -1$ and $S > 0$, these two expressions are not defined properly because of the term of $(1+\omega)^{\frac{1+\omega}{\omega}}$. For example, we have $(-0.1)^{1/11}$ for $\omega = -1.1$, which does not provide a real value. As is shown Fig. 1, $(1+\omega)^{\frac{1+\omega}{\omega}}$ is not available for $\omega < -1$ and it has a limiting value 1 for the vacuum state of $\omega = -1$. This arises because we introduce the first- and second-law with the integrability condition to describe the FRW universe with dark fluid in terms of a thermodynamic system, in addition the conservation law. We note that for the phantom regime, the conservation law of Eq.(8) is not compatible with the integrability condition of Eqs.(16) and (18).

It is suggested that if the temperature of a dark energy fluid is negative, the phantom regime ($\omega < -1$) is thermodynamically forbidden, even though its entropy is positive.

However, we note that this statement may be changed when one chooses specific forms of $C(\omega)$ and $\tilde{C}(\omega)$. For example, one may take $C = (1+w)^{2n-(1+w)/w}$ and $\tilde{C} = C/w$ with $n = 0, 1, 2, \cdots$[3]. It seems to appear a little more reasonable because as one approaches the big rip, one would expect a given quantization characterized by n to take place. In this case, a phantom phase may be allowed because $p(T)/\rho(T) = \omega$ provides a correct equation of state for $\omega < -1$.

The vacuum state of $\omega = -1$ corresponds to the marginal case: The total energy $E = \rho V$ increases during the expansion, while its energy density is constant [5]. In this case, if entropy is zero, the temperature is ill-defined. The reverse case is also possible to occur. This means that the thermodynamic interpretation for the vacuum state is still obscure.

Finally, one may introduce a negative chemical potential to avoid negative temperature [10, 11]. In this case, both the entropy and temperature of the phantom fluid may be positive when the chemical potential is negative. However, there exist controversial issues to find the form of chemical potential for the FRW universe [12] and the accretion of phantom fields by black holes [13, 14].

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[1] R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. **91** (2003) 071301 [arXiv:astro-ph/0302506]; J. A. S. Lima, J. V. Cunha and J. S. Alcaniz, Phys. Rev. D **68** (2003) 023510 [arXiv:astro-ph/0303388];

[2] J. D. Barrow, Class. Quant. Grav. **21** (2004) L79 [arXiv:gr-qc/0403084]; S. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D **71** (2005) 063004 [arXiv:hep-th/0501025].

[3] S. Nesseris and L. Perivolaropoulos, Phys. Rev. D **70** (2004) 123529 [arXiv:astro-ph/0410309].

[4] J. A. S. Lima and A. . J. Maia, Phys. Rev. D **52** (1995) 5628 [arXiv:gr-qc/9505052]; J. A. E. Carrillo, J. A. S. Lima and A. . J. Maia, Int. J. Theor. Phys. **35** (1996) 2013 [arXiv:hep-th/9906016].

[5] J. A. S. Lima and J. S. Alcaniz, Phys. Lett. B **600** (2004) 191 [arXiv:astro-ph/0402265].

[6] P. F. Gonzalez-Diaz and C. L. Siguenza, Nucl. Phys. B **697** (2004) 363. [arXiv:astro-ph/0407421]; P. F. Gonzalez-Diaz and C. L. Siguenza, Phys. Lett. B **589** (2004) 78.

[7] E. W. Kolb and M. S. Turner, *The Early Universe*, Addison-Wesley (1990), p.65.

[8] D. Youm, Phys. Lett. B **531** (2002) 276 [arXiv:hep-th/0201268].

[9] E. P. Verlinde, arXiv:hep-th/0008140;

I. Savonije and E. P. Verlinde, Phys. Lett. B **507** (2001) 305 [arXiv:hep-th/0102042].

R. G. Cai, Y. S. Myung and N. Ohta, Class. Quant. Grav. **18** (2001) 5429 [arXiv:hep-th/0105070]; S. Nojiri, S. D. Odintsov and S. Ogushi, Int. J. Mod. Phys. A **16** (2001) 5085 [arXiv:hep-th/0105117]; R. G. Cai and Y. S. Myung, Phys. Rev. D **67** (2003) 124021 [arXiv:hep-th/0210272]. Y. S. Myung, Phys. Lett. B **578** (2004) 7 [arXiv:hep-th/0306180]; M. R. Setare, Phys. Lett. B **602** (2004) 1 [arXiv:hep-th/0409055].

[10] S. H. Pereira and J. A. S. Lima, arXiv:0806.0682 [astro-ph], to appear in PLB.

[11] S. H. Pereira, arXiv:0806.3701 [astro-ph].

[12] J. A. d. Pacheco, arXiv:0808.1863 [astro-ph].

[13] J. A. de Freitas Pacheco and J. E. Horvath, Class. Quant. Grav. **24** (2007) 5427 [arXiv:0709.1240 [gr-qc]].

[14] J. A. S. Lima, S. H. Pereira, J. E. Horvath and D. C. Guariento, arXiv:0808.0860 [gr-qc].